General Relativity

Summary/Overview

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1 Lorentz-Transformations

1.1 Postulates of Special Relativity

Einstein came up with two postulates in 1905:

P1: Principle of relativity between inertial coordinate systems (just as for Galileo transformations)

P2: Constancy of the speed of light in vacuum, irrespective of the motion of the source or detector

Specifically, P1 and P2 follow from the statement:

The laws of nature are invariant under Lorentz-transformations.

1.2 Lorentz-Transformations

Lorentz-transformations act between two space-time cartesian coordinate systems:

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + a^{\alpha}, \qquad \alpha = 1, 2, 3, 0,$$

where $\Lambda^{\alpha}{}_{\beta}$ and a^{α} are constants (they become space-time dependent in general relativity, which makes the Lorentz-transformations non-linear) and the $\Lambda^{\alpha}{}_{\beta}$ obey

$$\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta} = \mathrm{diag}(1, 1, 1, -1).$$

PROPER TIME:

With this property, the proper time

$$d\tau^2 := dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = -\eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

is Lorentz-invariant, since

$$d\tau'^2 = -\eta_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = -\underbrace{\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta}}_{=\eta_{\gamma\delta}} dx^{\gamma} dx^{\delta} = d\tau^2.$$

MICHELSON-MORLEY EXPERIMENT:

This explains the Michelson-Morley experiment 1878. They measured

$$\frac{|c_{\parallel} - c_{\perp}|}{|c_{\parallel} + c_{\perp}|} \leq 3 \cdot 10^{-10}.$$

The earth's relative velocity is $v_{\oplus}/c \sim 10^{-4}$ and from thus it follows that one would theoretically (classically) expect $|c_{\parallel} - c_{\perp}|/|c_{\parallel} + c_{\perp}| = (v_{\oplus}/c)^2 \sim 10^{-8} \gg 3 \cdot 10^{-10}$. However, from $d\tau^2 = d\tau'^2$ follows actually $|c_{\parallel} - c_{\perp}|/|c_{\parallel} + c_{\perp}| = 0$. That is, because if the light wave in one system propagates with

$$\frac{\sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}}{dt} = \left|\frac{d\vec{x}}{dt}\right| = 1 \,(\cdot\, c)$$

$$\Leftrightarrow \quad 0 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = d\tau^2$$

$$= d\tau'^2 = dt'^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2$$

$$\Leftrightarrow \quad 1 = \frac{\sqrt{(dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2}}{dt'} = \left|\frac{d\vec{x}'}{dt'}\right| = \left|\frac{d\vec{x}}{dt}\right|.$$

TRANSFORMATION MATRIX:

For $\vec{v} = v\hat{e}_x$ one might guess

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \quad \Rightarrow \quad x' = \Lambda x = \begin{pmatrix} \gamma(x_1 + vt) \\ x_2 \\ x_3 \\ \gamma(t + v x_1) \end{pmatrix}$$

since this yields Galilei transformation for $\gamma \to 1$. So it should be $\gamma \to 1$ for $v \to 0$. Confirm, that

$$\Lambda^{\alpha}{}_{\gamma} \eta_{\alpha\beta} \Lambda^{\beta}{}_{\delta} = \eta_{\gamma\delta} \quad \Leftrightarrow \quad \Lambda^T \eta \Lambda = \eta.$$

For that to be the case, it has to be $\gamma = 1/\sqrt{1 - v^2}$.

1.3 Lorentz Groups

All the transformations

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + a^{\alpha}$$

form the Poincaré group. The subgroup with $a^{\alpha} = 0$ forms the homogenous Lorentz group. The sub-subgroup with $\Lambda^0_0 \geq 1$ and $\det \Lambda = 1$ is called proper Lorentz group, which is a Lie group and hence can be constructed with generators. Three generators for rotations (Euler angles) and three for boosts (components of $\vec{v}$).

The improper Lorentz group (where $\Lambda^0_0 < 1$ or $\det \Lambda \neq 1$) essentially consist of space and time reflections.

1.4 Lorentz Tensors

Lorentz tensors are quantities, which transform homogenously under Lorentz transformations:

scalars:	$s'(x') = s(x),$
contravar. 4-vectors:	$V'^{\alpha}(x') = \Lambda^{\alpha}{}_{\beta} V^{\beta}(x),$
covar. 4-vectors:	$W'_{\alpha}(x') = \Lambda_{\alpha}^{\ \beta} W_{\beta}(x),$

where $\Lambda_{\alpha}^{\ \beta} = \eta_{\alpha\gamma} \Lambda^{\gamma}{}_{\delta} \eta^{\delta\beta}$.

New Lorentz tensors can be built from old ones (R, S) by

linear combination:	$T^{\alpha}{}_{\beta} = aR^{\alpha}{}_{\beta} + bS^{\alpha}{}_{\beta}$
direct product:	$T^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = R^{\alpha}{}_{\beta} S^{\gamma}{}_{\delta}$
contraction:	$T = R^{\alpha}{}_{\alpha}$

Tensor transform like

$$T'^{\alpha_1\alpha_2\dots\alpha_n}{}_{\beta_1\beta_2\dots\beta_m} = \Lambda^{\alpha_1}{}_{\mu_1} \Lambda^{\alpha_2}{}_{\mu_2} \cdots \Lambda^{\alpha_n}{}_{\mu_n} \quad \Lambda_{\beta_1}{}^{\nu_1} \Lambda_{\beta_2}{}^{\nu_2} \cdots \Lambda_{\beta_n}{}^{\nu_n} \quad T^{\mu_1\mu_2\dots\mu_n}{}_{\nu_1\nu_2\dots\nu_m}.$$

The components of the Minkowski-Tensor, the Levi-Civita-Symbol and the zero tensor (all components zero) are independent from the inertial frame.

2 Particle Dynamics

2.1 4-Momentum

RELATIVISTIC EQUATION OF MOTION:

Suppose in the rest frame the force $\vec{F}$ is known and Newtonian dynamics $\vec{F} = \dot{\vec{p}}$ holds. Then, a relativistic equation for a particle with rest mass m and coordinates $x^\alpha(\tau)$ could be

$$\frac{dp^\alpha}{d\tau} := m\frac{d^2x^\alpha}{d\tau^2} = f^\alpha.$$

m and τ are scalars, x^α is a Lorentz vector. Thus, if f^α is also a Lorentz vector, this holds in any frame. In the rest frame, where $dt = d\tau$, it is

$$f^\alpha_{\text{rest}} = (\vec{F}, 0),$$

in an arbitrary frame it is

$$f^\alpha = \Lambda^\alpha{}_\beta(\vec{v})\, f^\beta_{\text{rest}} = \left(\vec{F} + (\gamma - 1)\, \hat{v}(\hat{v}\vec{F}), v(\vec{v}\vec{F})\right),$$

as follows from the Lorentz transformation for an arbitrary $\vec{v}$.

COMPONENTS OF THE 4-MOMENTUM:

Using (as known from 1.2)

$$d\tau = \sqrt{dt^2 - d\vec{x}^2} = dt\sqrt{1 - v} = dt/\gamma,$$

which is just time dilation, to get the components of 4-momentum:

$$p^\mu = m\frac{dx^\alpha}{d\tau} = \begin{pmatrix} \gamma m\, d\vec{x}/dt \\ \gamma m\, dt/dt \end{pmatrix} = \begin{pmatrix} \gamma m\vec{v} \\ \gamma m \end{pmatrix} = \begin{pmatrix} \vec{p} \\ E \end{pmatrix}.$$

Since

$$E = \frac{m}{\sqrt{1 - v^2/c^2}} \approx mc^2 + \frac{m}{2}v^2 + \cdots,$$

E is obviously the energy.

CONSERVATION OF 4-MOMENTUM:

If non-relativistically the energy and momentum are conserved, then the 4-momentum is also conserved

$$\sum_i q_i^\alpha - \sum_i p_i^\alpha = 0,$$

for incoming particles q_i^α and outgoing ones p_i^α.

ENERGY-MOMENTUM-RELATION:

The equation

$$E^2 = \vec{p}^2 + m^2$$

holds, since (using $d\tau^2 = -\eta_{\alpha\beta}dx^\alpha dx^\beta$)

$$\vec{p}^2 - E^2 = \eta_{\alpha\beta}p^\alpha p^\beta = m^2\eta_{\alpha\beta}\frac{dx^\alpha dx^\beta}{d\tau^2} = -m^2\frac{d\tau^2}{d\tau^2} = -m^2.$$

2.2 Electro-Magnetism

4-CURRENT:

Charged particles (charges e_n, positions $\vec{x}_n(t)$) generate the 4-current

$$J^\alpha(x) = \left(\vec{J}(x), \rho(x)\right) := \sum_n e_n\delta\left(\vec{x} - \vec{x}_n(t)\right)\frac{dx_n^\alpha}{dt}.$$

This doesn't look like a 4-vector, however it actually is, since

$$\begin{aligned} J^\alpha(x) &= \sum_n \int dt' e_n \underbrace{\delta\left(x - x_n(t')\right)}_{=\delta(\vec{x}-\vec{x}_n)\delta(t-t')}\frac{dx_n^\alpha(t')}{dt'} \\ &= \sum_n \int d\tau_n\, e_n\delta\left(x - x_n(\tau_n)\right)\frac{dx_n^\alpha(\tau_n)}{d\tau_n}, \end{aligned}$$

where the renaming $t' \to \tau_n$ was allowed since it's only an integration variable. In this form, it is obvious, that J^α is a 4-vector. Using this current, charge conservation $\partial_t\rho = -\nabla\vec{J}$ can be written in the Lorentz invariant form

$$\partial_\alpha J^\alpha(x) = 0.$$

MAXWELL-EQUATIONS:

In relativistic notation, the Maxwell equations read

$$\partial_\alpha F^{\alpha\beta} = -J^\beta, \qquad \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0,$$

where $F^{\alpha\beta}$ is the anti-symmetric field strength tensor. Classically, the electric and magnetic fields are used instead of $F^{\alpha\beta}$, but this is just a matter of notation/interpretation:

$$F^{ij} = \epsilon_{ijk}B_k, \qquad F^{0i} = E_i.$$

The electro-magnetic force on a particle with charge e is in terms of $F^{\alpha\beta}$

$$f^\alpha_{\text{EM}} = eF^\alpha{}_\beta\frac{dx^\beta}{d\tau}.$$

In the particle's rest frame that yields

$$f^\alpha_{\text{EM}} = eF^\alpha{}_\beta\delta^\beta{}_0 = eF^\alpha{}_0 \quad\Longrightarrow\quad \begin{cases} f^i = eF^i{}_0 = -eF^{i0} = eE_i \\ f^0 = eF^0{}_0 = -eF^{00} = 0 \end{cases}.$$

2.3 Energy-Momentum Tensor

The energy-momentum is defined quite analogous as the 4-current:

$$\mathcal{T}^{\alpha\beta}(x) := \sum_n p_n^\alpha(t)\frac{dx_n^\beta}{dt}\delta\left(\vec{x} - \vec{x}_n(t)\right), \quad \mathcal{T}^{\alpha\beta} = \mathcal{T}^{\beta\alpha}.$$

Whereas the 4-current describes densities and currents of charges, the energy-momentum tensor describes densities and currents of energy-momentum p^α, a 4-vector, and hence has a second index α. Just as for the 4-current one can proof that $\mathcal{T}^{\alpha\beta}$ is indeed a tensor and with $p^\alpha = m\,\partial_\tau x^\alpha$ it follows that $\mathcal{T}^{\alpha\beta}$ is symmetric. Using $f^\alpha = \partial_\tau p^\alpha$ yields

$$\frac{\partial}{\partial x^\beta}\mathcal{T}^{\alpha\beta}(x) := \sum_n \frac{d\tau}{dt}f_n^\alpha(t)\,\delta\left(\vec{x} - \vec{x}_n(t)\right).$$

For either free particles or particles with localized interactions (like hard collision) there are conservation laws:

$$\frac{\partial}{\partial x^\beta}\mathcal{T}^{\alpha\beta} = 0.$$

Not so, if there are long-range interactions. To rescue energy-momentum conservation, one introduces force fields so that the total energy-momentum tensor is conserved:

$$\frac{\partial}{\partial x^\beta}\mathcal{T}^{\alpha\beta}_{\text{tot}} = \frac{\partial}{\partial x^\beta}\left(\mathcal{T}^{\alpha\beta}_{\text{particles}} + \mathcal{T}^{\alpha\beta}_{\text{fields}}\right),$$

where $\mathcal{T}_{\text{particles}}$ is the $\mathcal{T}$ as defined above. For example, the long-range electro-magnetic force yields

$$\mathcal{T}^{\alpha\beta}_{\text{EM-fields}} = F^{\alpha\gamma}\eta_{\gamma\delta}F^{\beta\delta} - \frac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\alpha\delta},$$

which has the components

$$\mathcal{T}^{00}_{\text{EM}} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \qquad \text{(densitiy of field energy)},$$

$$\mathcal{T}^{i0}_{\text{EM}} = \left(\vec{E} \times \vec{B}\right)^i \qquad \text{(Poynting-Vector)}.$$

3 Principles of General Relativity

3.1 Einstein Principle of Equivalence (EEP)

THE EEP:

Einstein postulated the following EEP:

In an arbitrary gravitational field, it is possible to choose a locally inertial coordinate system (LICS), so that in a sufficient small region of space-time the laws of nature take the same form as in the absence of gravity (i. e. special relativity holds in the LICS).

EQUALITY OF INERTIAL AND GRAVITATIONAL MASS:

Consider a non-relativistic particle $\vec{x}_p$ and N other such particles $\vec{x}_n$ with a known Force $\vec{F}(\vec{x}_p - \vec{x}_n)$ (i. e. electrostatic) within a static, homogeneous gravitational field $\vec{g} = \text{const}$. The equation of motion for the particle $\vec{x}_p$ then reads

$$m_i \frac{d^2\vec{x}_p}{dt^2} = m_g\vec{g} + \sum_{n=1}^{N} \vec{F}(\vec{x}_p - \vec{x}_n),$$

where it was distinguished between the inertial and gravitational mass. After a transformation $\vec{x}' = \vec{x} + \vec{g}t/2, t' = t$, this equation looks as follows:

$$m_i \frac{d^2\vec{x}_p'}{dt^2} = \sum_{n=1}^{N} \vec{F}(\vec{x}_p' - \vec{x}_n') + (m_g - m_i)\vec{g}.$$

Obviously, if $m_g = m_i$ the observer in O' sees the same physics as the one in O just without gravitational field! That's just what the EEP states.

PARTICLE IN AN EXTERNAL GRAVITATIONAL FIELD:

Now, the equation of motion of a particle in an external gravitational field is derived. In the freely falling LICS, the equation of motion is given by, according to the EEP,

$$d^2\xi^\alpha/d\tau^2 = 0$$

with the proper time

$$d\tau^2 = -\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta,$$

just as in 1.2. In arbitrary coordinates x^μ this yields

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\mu}{\partial\tau}.$$

Contracting this equation with $\partial x^\lambda/\partial\xi^\alpha$ yields

$$0 = \delta^\lambda_\mu\frac{\partial^2 x^\mu}{\partial\tau^2} + \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\mu}{\partial\tau} = \frac{\partial^2 x^\lambda}{\partial\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau},$$

where the "affine connection" (which is no tensor!) is given as

$$\Gamma^\lambda{}_{\mu\nu} := \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}.$$

The proper time in terms of the new coordinates is given as

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}dx^\mu dx^\nu = -g_{\mu\nu}dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is the metric tensor, which is defined as

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}.$$

For $\Gamma^\lambda{}_{\mu\nu} = 0$, the particle does not feel a gravitational force.

THE AFFINE CONNECTION IN TERMS OF THE METRIC:

Take the derivative of the equation for the metric above:

$$\frac{\partial}{\partial x^\lambda}g_{\mu\nu} = \frac{\partial^2\xi^\alpha}{\partial x^\lambda\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu}\eta_{\alpha\beta}$$
$$= \Gamma^\kappa{}_{\lambda\mu}\frac{\partial\xi^\alpha}{\partial x^\kappa}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\kappa{}_{\lambda\nu}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\kappa}\eta_{\alpha\beta} = \Gamma^\kappa{}_{\lambda\mu}g_{\kappa\nu} + \Gamma^\kappa{}_{\lambda\nu}g_{\mu\kappa}.$$

Adding/subtracting the same formula with interchanged indices $\mu \leftrightarrow \lambda$ and $\nu \leftrightarrow \lambda$ yields

$$\frac{\partial}{\partial x^\lambda}g_{\mu\nu} + \frac{\partial}{\partial x^\mu}g_{\lambda\nu} - \frac{\partial}{\partial x^\nu}g_{\mu\lambda} = \cdots = 2\Gamma^\kappa{}_{\mu\lambda}g_{\nu\kappa}.$$

Define the inverse metric as $g^{\mu\nu}$ such that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$. The contraction of the equation with $g^{\nu\sigma}$ yields

$$\Gamma^\sigma{}_{\lambda\mu} = \frac{1}{2}g^{\sigma\nu}\left(\frac{\partial}{\partial x^\lambda}g_{\mu\nu} + \frac{\partial}{\partial x^\mu}g_{\lambda\nu} - \frac{\partial}{\partial x^\nu}g_{\mu\lambda}\right).$$

3.2 Newtonian Limit

Consider a non-relativistic particle, that is to say

$$\frac{dx}{d\tau} \ll \frac{dt}{d\tau},$$

we is supposed to obey the Newtonian equation of motion

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\phi_N, \qquad \phi_N = -\frac{GM}{r},$$

where the mass M is considered to be small.

For a non-relativistic particle in a week and static gravitational field the relativistic equation of motion of 3.1 reads

$$0 = \frac{\partial^2 x^\eta}{\partial\tau^2} + \Gamma^\eta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \approx \frac{\partial^2 x^\eta}{\partial\tau^2} + \Gamma^\eta{}_{00}\left(\frac{dt}{d\tau}\right)^2,$$

where

$$\Gamma^\eta{}_{00} = \frac{1}{2}g^{\eta\nu}(\partial_0 g_{0\nu} + \partial_0 g_{\nu 0} - \partial_\nu g_{00}) = -\frac{1}{2}g^{\eta\nu}\partial_\nu g_{00},$$

where the time derivatives vanish since the field was assumed to be static. Since the field is also assumed to be small, the metric deviates only a little from the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \qquad \Rightarrow \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}.$$

Thus, the spatial components of the equations of motion become

$$\frac{\partial^2 x^i}{\partial\tau^2} \approx \frac{1}{2}\eta^{i\nu}\left(\frac{dt}{d\tau}\right)^2 \partial_\nu h_{00} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla_i h_{00},$$

the time component on the other hand

$$\frac{\partial^2 t}{\partial\tau^2} \approx \frac{1}{2}\eta^{0\nu}\left(\frac{dt}{d\tau}\right)^2 \partial_\nu g_{00} = 0,$$

since the field is static and thus $\partial_0 g_{00} = 0$. Thus, take the constant $\partial t/\partial\tau$ to be 1, which yields

$$\frac{\partial^2\vec{x}}{\partial\tau^2} \approx \frac{1}{2}\nabla h_{00}.$$

Thus, for $h_{00} = -2\phi_N$ (thus $g_{00} = -1 - 2\phi_N$) the Newtonian equation is recovered.

3.3 Gravitational Redshift

Consider two identical clocks that send out light at the same frequency $\nu = 1/\Delta t$. If they are at rest ($dx^i = 0$), the space-time interval between two ticks Δs equals the time-interval:

$$\Delta s = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{-g_{00}(x_i)}dt_i = \Delta t$$
$$\Leftrightarrow \quad \sqrt{-g_{00}(x_1)}dt_1 = \sqrt{-g_{00}(x_2)}dt_2$$
$$\Rightarrow \quad \frac{\nu_2}{\nu_1} = \frac{dt_1}{dt_2} = \frac{\sqrt{-g_{00}(x_2)}}{\sqrt{-g_{00}(x_1)}}.$$

Calling $\nu_1 = \nu$ and $\nu_2 = \nu + \Delta\nu$ and using $g_{00} = -1 - 2\phi$, which was the result of 3.2 for week fields, yields

$$\frac{\Delta\nu}{\nu} = \frac{\nu_2 - \nu_1}{\nu_1} = \frac{\nu_2}{\nu_1} - 1 = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} - 1 = \sqrt{\frac{1 + 2\phi(x_2)}{1 + 2\phi(x_1)}} - 1$$
$$\approx (1 + \phi(x_2))(1 - \phi(x_1)) - 1 = \phi(x_2) - \phi(x_1) + \mathcal{O}(\phi^2)$$
$$\approx \Delta\phi.$$

4 Tensors and Curvatures

4.1 Tensors and the EEP

The EEP can be rephrased as follows:

A physical equation holds in a general gravitational field, if

1. *the equation is generally covariant (meaning the form is preserved under a general coordinate transformation) and*
2. *the equation holds in the absence of gravity (meaning for $g_{\mu\nu} = \eta_{\mu\nu}$, ${\Gamma^\lambda}_{\mu\nu} = 0$ special relativity holds locally).*

Such physical equations can be built out of tensors, which transform homogeneously under general coordinate transformations:

$$T'^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = \frac{dx'^{\mu_1}}{dx^{\eta_1}}\cdots\frac{dx'^{\mu_n}}{dx^{\eta_N}}\frac{dx^{\kappa_1}}{dx'^{\nu_1}}\cdots\frac{dx^{\kappa_n}}{dx'^{\nu_N}}T^{\eta_1\cdots\eta_n}_{\kappa_1\cdots\kappa_m}(x).$$

4.2 Transformation of the Metric and the Connection

THE METRIC:

The metric *does* transform like a tensor under a transformation $x \to x'$ (see definition from 3.1):

$$g'_{\mu\nu} = \eta_{\alpha\beta}\frac{d\xi^\alpha}{dx'^\mu}\frac{d\xi^\beta}{dx'^\nu} = \eta_{\alpha\beta}\frac{\partial x^\eta}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu}\frac{d\xi^\alpha}{dx^\eta}\frac{d\xi^\beta}{dx^\kappa} = g_{\eta\kappa}\frac{\partial x^\eta}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu}.$$

THE AFFINE CONNECTION:

The affine connection *does not* transform like a tensor! Instead, the transformed affine connection reads

$${\Gamma'^\lambda}_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}{\Gamma^\rho}_{\sigma\tau} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}.$$

4.3 The Covariant Derivative

Consider a vector V^μ. Under arbitrary coordinate transformation,

$$\partial_\nu V_\mu = \frac{\partial}{\partial x^\nu}V_\mu \coloneqq V_{\mu,\nu}$$

is *not* a tensor:

$$V'_\mu = \frac{\partial x^\sigma}{\partial x'^\mu}V_\sigma$$

$$\Rightarrow \quad \frac{\partial}{\partial x'^\nu}V'_\mu = \frac{\partial}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}V_\sigma = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial}{\partial x'^\nu}V_\sigma + \left(\frac{\partial}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\mu}\right)V_\sigma$$

$$= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\frac{\partial}{\partial x^\tau}V_\sigma + \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}V_\sigma.$$

This structure is very similar as for the affine connection. Thus,

$$\boxed{V_{\mu;\nu} \coloneqq V_{\mu,\nu} - {\Gamma^\lambda}_{\mu\nu}V_\lambda, \qquad {V^\mu}_{;\nu} \coloneqq {V^\mu}_{,\nu} + {\Gamma^\mu}_{\lambda\nu}V^\lambda}$$

are tensors, since

$$V'_{\mu;\nu} = V'_{\mu,\nu} - {\Gamma'^\lambda}_{\mu\nu}V'_\lambda = \cdots = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}V_{\sigma;\tau}.$$

The covariant derivative of an rank-2 tensor then reads

$${T^\mu}_{\nu;\lambda} = {T^\mu}_{\nu,\lambda} + {\Gamma^\mu}_{\lambda\kappa}{T^\kappa}_\nu - {\Gamma^\kappa}_{\nu\lambda}{T^\mu}_\kappa,$$

where ${T^\mu}_{\nu,\lambda} \coloneqq \partial_\lambda {T^\mu}_\nu$. The covariant derivative acts as a linear operator, just as the ordinary derivative.

4.4 Electro-Magnetism in Curved Space-Time

GENERAL RECIPE:

1. Find the appropriate special relativity equations, holding in the absence of gravity.
2. Replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and ordinary by covariant derivatives.

Then, the resulting equations also hold in the presence of a gravitational field.

COVARIANT MAXWELL EQUATIONS:

The covariant Maxwell equations read (compare with 2.2):

$${F^{\mu\nu}}_{;\mu} = -J^\nu, \qquad F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0.$$

The force 4-vector stays the same:

$$f^\mu = e{F^\mu}_\nu\frac{dx^\nu}{d\tau}.$$

4.5 The Riemann Tensor and the Ricci Tensor

RIEMANN TENSOR:

The first (covariant) derivative of the metric, $g_{\mu\nu;\lambda}$, ist of course a tensor, but one can show easily with the definition of the affine connection (which sits inside the covariant derivative) that

$$g_{\mu\nu;\lambda} = 0.$$

It has been shown that the only tensor linear in the second order derivative of the metric is the Riemann tensor:

$${R^\lambda}_{\mu\nu\rho} \coloneqq {\Gamma^\lambda}_{\mu\nu,\rho} - {\Gamma^\lambda}_{\mu\rho,\nu} + {\Gamma^\sigma}_{\mu\nu}{\Gamma^\lambda}_{\rho\sigma} - {\Gamma^\sigma}_{\mu\rho}{\Gamma^\lambda}_{\nu\sigma},$$

which also can be written as

$$R_{\lambda\mu\nu\rho} = \frac{1}{2}\left(g_{\lambda\nu,\mu,\rho} - g_{\mu\nu,\lambda,\rho} - g_{\lambda\rho,\mu,\nu} + g_{\mu\rho,\lambda,\nu}\right) + g_{\kappa\sigma}\left({\Gamma^\kappa}_{\nu\lambda}{\Gamma^\sigma}_{\mu\rho} - {\Gamma^\kappa}_{\rho\lambda}{\Gamma^\sigma}_{\mu\nu}\right).$$

Note, that the usual convention has an opposite overall sign!

It has $4^4 = 256$ components, but due to symmetries, only 20 are independent:

$$R_{\lambda\mu\,\nu\rho} = R_{\nu\rho\,\lambda\mu}$$
$$R_{\lambda\mu\,\nu\rho} = -R_{\mu\lambda\,\nu\rho} = -R_{\lambda\mu\,\rho\nu}$$
$$R_{\lambda\,\mu\nu\rho} + R_{\lambda\,\rho\mu\nu} + R_{\lambda\,\nu\rho\mu} = 0 \qquad \text{(first/algebraic B.I.)}$$
$$R_{\lambda\mu\,\nu\rho;\sigma} + R_{\lambda\mu\,\sigma\nu;\rho} + R_{\lambda\mu\,\rho\sigma;\nu} = 0 \qquad \text{(second/differential B.I.)}$$

Here, B.I. is short-hand for *Bianchi identity*. One may check all this in the LICS, where $\Gamma = 0$.

RICCI TENSOR AND RICCI SCALAR:

The Ricci tensor is defined as

$$R_{\mu\rho} \coloneqq {R^\lambda}_{\mu\lambda\rho},$$

the Ricci scalar as

$$R \coloneqq g^{\mu\nu}R_{\mu\nu} = {R^\mu}_\mu.$$

The Ricci tensor is symmetric:

$$R_{\mu\rho} = {R^\lambda}_{\mu\lambda\rho} = g^{\lambda\kappa}R_{\kappa\mu\lambda\rho} = g^{\lambda\kappa}R_{\lambda\rho\kappa\mu} = {R^\kappa}_{\rho\kappa\mu} = R_{\rho\mu}.$$

Also, other contractions of the Riemann tensor can be given as the Ricci tensor, e.g.

$${R^\lambda}_{\rho\mu\lambda} = g^{\lambda\nu}R_{\nu\rho\mu\lambda} = g^{\lambda\nu}R_{\mu\lambda\nu\rho} = -g^{\lambda\nu}R_{\lambda\mu\nu\rho} = -{R^\nu}_{\mu\nu\rho} = -R_{\mu\rho}.$$

Finally, it holds that

$$\frac{1}{\sqrt{g}}\epsilon^{\lambda\mu\nu\rho}R_{\lambda\mu\nu\rho} = 0, \qquad g \coloneqq -\det g_{\mu\nu}.$$

THE CONTRACTED BIANCHI IDENTITY:

Using ${g^{\nu\lambda}}_{;\sigma} = 0$ from the very top of 4.5, it follows that

$$g^{\nu\lambda}R_{\lambda\mu\nu\rho;\sigma} = \left(g^{\nu\lambda}R_{\lambda\mu\nu\rho}\right)_{;\sigma} = {R^\nu}_{\mu\nu\rho;\sigma} = R_{\mu\rho;\sigma} \quad \text{and}$$

$$g^{\mu\rho}{R^\lambda}_{\mu\rho\sigma;\lambda} = g^{\mu\rho}g^{\lambda\kappa}R_{\kappa\mu\rho\sigma;\lambda} = -g^{\mu\rho}g^{\lambda\kappa}R_{\mu\kappa\rho\sigma;\lambda}$$
$$= -g^{\lambda\kappa}{R^\rho}_{\kappa\rho\sigma;\lambda} = -g^{\lambda\kappa}R_{\kappa\sigma;\lambda} = -{R^\lambda}_{\sigma;\lambda}$$

Thus, the following form of the second Bianchi identity yields

$$R_{\lambda\mu\nu\rho;\sigma} - R_{\lambda\mu\nu\sigma;\rho} + R_{\lambda\mu\rho\sigma;\nu} = 0 \qquad |g^{\lambda\nu}\,\cdot$$
$$\Leftrightarrow \quad R_{\mu\rho;\sigma} - R_{\mu\sigma;\rho} + {R^\nu}_{\mu\rho\sigma;\nu} = 0 \qquad |g^{\mu\rho}\,\cdot$$
$$\Leftrightarrow \quad R_{;\sigma} - {R^\rho}_{\sigma;\rho} - {R^\lambda}_{\sigma;\lambda} = R_{;\sigma} - 2{R^\mu}_{\sigma;\mu} = 0$$
$$\Leftrightarrow \quad {R^\mu}_{\sigma;\mu} - \tfrac{1}{2}\delta^\mu_\sigma R_{;\mu} = \left({R^\mu}_\sigma - \tfrac{1}{2}\delta^\mu_\sigma R\right)_{;\mu} = 0 \qquad |g^{\nu\sigma}\,\cdot$$
$$\Leftrightarrow \quad \boxed{\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0}$$

4.6 Covariant Derivative along a Curve

The covariant derivative of a vector $s_\mu(\tau)$ along a curve $x^\mu(\tau)$ is

$$\boxed{\frac{Ds_\mu}{D\tau} \coloneqq \frac{ds_\mu}{d\tau} - {\Gamma^\lambda}_{\mu\nu}\frac{dx^\nu}{d\tau}s_\lambda,}$$

since $Ds_\mu/D\tau$ behaves like a tensor under coordinate transformation, as can be shown straight forwardly by plugging in the transformation properties of s'_μ and ${\Gamma'^\lambda}_{\mu\nu}$. In the absence of other forces, for the LICS ($\Gamma = ds/d\tau = 0$) holds that

$$\frac{Ds_\mu}{d\tau} = 0,$$

and since $Ds_\mu/d\tau$ is invariant this also holds in any other frame.

4.7 Parallel Transport

Say that any vector s_μ is parallel transported if it obeys

$$\frac{Ds_\mu}{D\tau} = 0 \quad \Leftrightarrow \quad \frac{ds_\mu}{d\tau} = \Gamma^\lambda{}_{\mu\nu}\frac{dx^\nu}{d\tau}s_\lambda.$$

Define the change of the vector s_μ after being parallel transported around a closed loop $\mathcal{C}$ (parametrized as $x^\mu(\tau)$) as

$$\Delta s_\mu(\mathcal{C}) \coloneqq s_\mu(\tau_{\text{end}}) - s_\mu(\tau_{\text{start}}).$$

Consider an arbitrary area $\mathcal{A}$ bounding $\mathcal{C}$ and break it up into $N \gg 1$ "tiles" $\delta\mathcal{A}_n$ with borders $\delta\mathcal{C}_n$. Thus it follows

$$\Delta s_\mu(\mathcal{C}) = \sum_{n=1}^{N} \Delta s_\mu(\mathcal{C}_n).$$

Now, $s_\mu(\mathcal{C}_n)$ needs to be evaluated. Starting from some point $P = x^\mu(\tau_{\text{start}})$ at some other point $x^\mu(\tau)$ the vector s_μ reads

$$s_\mu(\tau) = s_\mu(P) + \int_{\tau_{\text{start}}}^{\tau} d\tau\, \Gamma^\lambda{}_{\mu\nu}\frac{dx^\nu}{d\tau}s_\lambda,$$

which is derived from the condition $Ds_\mu/D\tau = 0$. Since the curve $\mathcal{C}_n$ is small, the integrand can be Taylor expanded about P:

$$\Gamma^\lambda{}_{\mu\nu}(\tau) \approx \Gamma^\lambda{}_{\mu\nu}(P) + \partial_\rho\Gamma^\lambda{}_{\mu\nu}(P)\big(x^\rho(\tau) - x^\rho(P)\big),$$
$$s_\lambda(\tau) \approx s_\lambda(P) + \Gamma^\sigma{}_{\lambda\rho}(P)s_\sigma(P)\big(x^\rho(\tau) - x^\rho(P)\big).$$

This yields up to terms of first order in $x^\nu(\tau) - x^\nu(P)$

$$\begin{aligned} s_\mu(\tau) &\\ &= s_\mu(P) + \Gamma^\lambda{}_{\mu\nu}(P)s_\lambda(P)\int_{\tau_{\text{start}}}^{\tau} d\tau\frac{dx^\nu}{d\tau} \\ &\quad + \Big(\Gamma^\lambda{}_{\mu\nu}(P)\Gamma^\sigma{}_{\lambda\rho}(P) + \partial_\rho\Gamma^\sigma{}_{\mu\nu}(P)\Big)s_\sigma(P) \\ &\quad \int_{\tau_{\text{start}}}^{\tau} d\tau\,\big(x^\rho(\tau) - x^\rho(P)\big)\frac{dx^\nu}{d\tau}. \end{aligned}$$

By definition it is $\Delta s_\mu(\mathcal{C}_n) = s_\mu(t_{\text{end}}) - s_\mu(P)$. For $\tau = \tau_{\text{end}}$ the integrals become closed loop integrals for which $\oint dx^\nu$ is always zero. Obviously, this holds for $\oint x^\rho(P)\,dx^\nu \sim \oint dx^\nu$. Thus,

$$\Delta s_\mu = \Big(\Gamma^\lambda{}_{\mu\nu}(P)\Gamma^\sigma{}_{\lambda\rho}(P) + \partial_\rho\Gamma^\sigma{}_{\mu\nu}(P)\Big)s_\sigma(P)\oint_{\mathcal{C}_n} dx^\nu x^\rho.$$

ν and ρ, being dummy indices, can be interchanged. Using

$$0 = \oint d(x^\nu x^\rho) = \oint(dx^\nu x^\rho + dx^\rho x^\nu)$$
$$\Leftrightarrow \oint dx^\nu x^\rho = -\oint dx^\rho x^\nu$$

yields

$$\begin{aligned} \Delta s_\mu &= \frac{1}{2}\Big(\Delta s_\mu + \Delta s_\mu(\rho \leftrightarrow \nu)\Big) \\ &= \frac{1}{2}(\Gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\lambda\rho} + \partial_\rho\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\sigma{}_{\lambda\nu} - \partial_\nu\Gamma^\sigma{}_{\mu\rho})s_\sigma\big|_P\oint_{\mathcal{C}_n} dx^\nu x^\rho \end{aligned}$$

and thus, using the definition of the Riemann tensor from 4.5,

$$\boxed{\Delta s_\mu(\mathcal{C}_n) = \frac{1}{2}R^\sigma{}_{\mu\nu\rho}(P)s_\sigma(P)\oint_{\mathcal{C}_n} dx^\nu x^\rho.}$$

Evidently, s_μ changes if and only if $R^\sigma{}_{\mu\nu\rho}(P) \neq 0$. Thus, the Riemann tensor indicates the presence of a genuine gravitational field instead of mere exotic coordinates (instead of the metric, which can also for a plane manifold look ugly in strange coordinates).

It can be proofed that for the metric $g_{\mu\nu}$ to be equivalent to the constant Minkowski metric *globally* (meaning there are coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ *globally*), the conditions are

1. $R^\lambda{}_{\mu\nu\rho}[g(x)] = 0\ \forall x$
2. at some point x the matrix $g_{\mu\nu}(x)$ has three positive and one negative eigenvalues.

4.8 Tensor Densities

DEFINITION:

A tensor density is an object which transforms almost like a tensor, specifically

$$\mathcal{T}'^{\cdots}_{\cdots} = \det{}^w\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]\cdot\frac{\partial x'^{\cdots}}{\partial x^{\cdots}}\cdots\frac{\partial x^{\cdots}}{\partial x'^{\cdots}}\cdots\mathcal{T}^{\cdots}_{\cdots},$$

where the $\cdots$ stand for all the indices. The determinant is the Jacobian determinant and w is called the *weight* of the tensor density.

METRIX DETERMINANT:

The determinant of the metric $g = -\det g_{\mu\nu}$ is a scalar density of weight $w = -2$, since (using $\det A^T = \det A$)

$$g' = -\det\frac{\partial x^\lambda}{\partial x'^\mu}g_{\lambda\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \det{}^2\frac{\partial x^\lambda}{\partial x'^\mu}g = \det{}^{-2}\frac{\partial x'^\mu}{\partial x^\lambda}g.$$

LEVI-CIVITA SYMBOL:

The Levi-Civita symbol is a tensor density with weight $w = -1$. A tensor is a tensor density with $w = 0$. It holds

$$\underbrace{\epsilon^{\lambda\mu\nu\rho}}_{w=-1}\underbrace{R_{\lambda\mu\nu\rho}}_{w=0}\underbrace{1/\sqrt{g}}_{w=1} = \underbrace{0}_{w=0}.$$

DELTA-FUNCTION:

$\sqrt{g}\,d^4x$ is a scalar volume element and thus $\delta^4(x)/\sqrt{g}$ is a scalar density because

$$\int d^4x\,\sqrt{g}\frac{1}{\sqrt{g}}\delta^4(x)f(x) = f(0).$$

4.9 Energy-Momentum Tensor in General Relativity

For the Special Relativity case, the energy-momentum tensor was given in 2.3 and it obeys

$$\hat{T}^{\mu\nu}{}_{,\mu}\big|_{\text{SR}} = 0.$$

Following the recipe in 4.4, there should be a tensor $T^{\mu\nu}$ with

$$T^{\mu\nu}{}_{;\mu} = 0,$$

which reduces to $\hat{T}^{\mu\nu}$ in a LICS. Similar to 2.3, the tensor reads

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{g}}\sum_n m_n\int dx_n^\mu\frac{dx_n^\nu}{d\tau_n}\delta(x - x_n).$$

5 Gravitational Field Equations

5.1 Derivation of the Einstein Equations

ANSATZ:
The electro-magnetic fields do not carry charges themselves which makes the Maxwell equations linear. However, gravitational fields *do* carry energy-momentum themselves and thus the differential equations are by necessity non-linear.
The Einstein Equations can be searched for being guided by the following principles: the EEP and the Newtonian limit.
In 3.2 it was found that $g_{00} = -1 - 2\phi$. Also, T_{00} corresponds to the energy density. Thus, the Poisson equation $\nabla^2\phi = 4\pi G_N \rho_{\text{mass}}$ may be rewritten as

$$\nabla^2 g_{00} = -8\pi G_N T_{00}.$$

This leads oneself to the Ansatz

$$G_{\mu\nu} = -8\pi G_N T_{\mu\nu},$$

where $G_{\mu\nu}$ must be a tensor, which

1. involves a second order derivative of the metric linearly,
2. is symmetric (since $T_{\mu\nu}$ is symmetric),
3. is conserved, i.e. $G^{\mu\nu}{}_{;\mu} = 0$ (since $T_{\mu\nu}$ is conserved) and
4. obeys $G_{00} \approx \nabla^2 g_{00}$ for weak static fields.

CONSTRUCTING $G_{\mu\nu}$:
The only tensor available is the Riemann tensor and since $G_{\mu\nu}$ has only two indices, it has to be built out of $R_{\mu\nu}$ and R:

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R.$$

This already fulfills condition 2. The 3. condition requires

$$G^{\mu\nu}{}_{;\mu} = (c_1 R^{\mu\nu} + c_2 g^{\mu\nu} R)_{;\mu} = 0.$$

Comparing this to the contracted Bianchi identity in 4.5 yields that $G^{\mu\nu}$ must be of the form

$$G^{\mu\nu} = c\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right).$$

CALCULATE THE FACTOR c:
c follows out of the 4. condition. Consider a non-relativistic system, for which $|T_{ij}| \ll |T_{00}|$ (low velocities). In this case,

$$G_{ij} = -8\pi G_N T_{ij} \approx 0 \quad \Leftrightarrow \quad R_{ij} \approx \frac{1}{2} g_{ij} R.$$

Weak fields (see condition 4) implies $g_{\mu\nu} \approx \eta_{\mu\nu}$ and therefore

$$R = g^{\mu\nu} R_{\mu\nu} \approx \textstyle\sum_i R_{ii} - R_{00} = \frac{1}{2}\sum_i \eta_{ii}\, R - R_{00} = \frac{3}{2} R - R_{00}$$
$$\Leftrightarrow \quad R = 2R_{00},$$

recalling the convention $\eta = \operatorname{diag}(-1, 1, 1, 1)$. Thus,

$$G_{00} = c\left(R_{00} - \tfrac{1}{2} g_{00} R\right) \approx 2cR_{00} \approx 2c\left(\textstyle\sum_i R_{i0i0} - R_{0000}\right).$$

For a weak static field in a LICS, it is $\Gamma \approx 0$ and therefore,

$$R_{\lambda\mu\nu\rho} \approx \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\rho} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\rho} - \frac{\partial^2 g_{\lambda\rho}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\rho}}{\partial x^\lambda \partial x^\nu}\right)$$
$$\Longrightarrow \quad R_{0000} \approx 0, \quad R_{i0j0} \approx \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j}.$$

The latter follows directly from the *static* field, which makes all time derivatives vanish. Plugging those results in yields

$$G_{00} \approx 2c \sum_i \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = c\nabla^2 g_{00}.$$

Comparing this to condition 4 directly yields $c = 1$.

THE EINSTEIN EQUATIONS:
Thus, the Einstein Field Equations (1915) read

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu},}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

is called the *Einstein tensor*.

5.2 Remarks to the Einstein Equations

CURVATURE IN VACUUM:
Consider a universe, where some areas are not filled with matter, but are in vacuum. In those areas holds $T_{\mu\nu} = 0$ and thus

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \qquad\qquad |g^{\mu\nu}\cdot$$
$$\Leftrightarrow \quad R - \frac{1}{2} R = 0 \quad \Leftrightarrow \quad R = 0 \quad \Longrightarrow \quad R_{\mu\nu} = 0.$$

Where $R = 0 \Longrightarrow R_{\mu\nu} = 0$ follows from the Einstein equations for $T_{\mu\nu} = 0$. In one or two spatial dimensions it also holds $R_{\mu\nu} = 0 \Longrightarrow R_{\mu\nu\lambda\rho} = 0$, i.e. space is never curved in vacuum areas. However, for three spatial dimensions this is not true and $R_{\mu\nu\lambda\rho}$ can be non-zero also in vacuum areas.

RICCI SCALAR AND ENERGY-MOMENTUM TENSOR:
Contracting the Einstein equation with $g^{\mu\nu}$ yields

$$R - 2R = -8\pi G_N T^\mu{}_\mu \quad \Leftrightarrow \quad R = 8\pi G_N T^\mu{}_\mu.$$

Thus, the Einstein equations may be given as

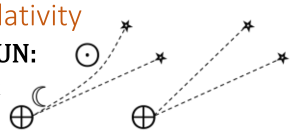
$$R_{\mu\nu} = -8\pi G_N\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda\right).$$

COSMOLOGICAL CONSTANT:
Adding a term $-\Lambda g_{\mu\nu}$ to $G_{\mu\nu}$, where Λ is a constant, yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}.$$

Actually, this contradicts condition 4 in 5.1, however, Λ might be small enough so that its impact on the Newtonian limit is negligible. In 1999 it was discovered that $\Lambda \sim 10^{-52}\ \text{m}^{-2}$.

5.3 Classical Tests of General Relativity



DEFLECTION OF THE LIGHT BY THE SUN:
At a solar eclipse in 1919, the apparent angle between two stars φ was measured, where the light of one of the stars passed the sun in a short distance (the solar eclipse was necessary to shield the sunlight and be able to see the starlight). Half a year later, the sun moved towards the other side of the earth and the angle was measured again (without needing the moon). General Relativity predicts that those measured angles differ by

$$\Delta\varphi = \frac{4G_N M_\odot}{R_\odot} \approx 1{,}75 \text{ arcsec} \approx 4{,}86 \cdot 10^{-4} \text{ degrees}.$$

Assigning a mass $m = E/c^2$ to the photons and applying Newton's theory only gives half of this value. Two independent groups conducted this experiment in 1919 and came out with

$$\Delta\varphi = (1{,}98 \pm 0{,}12) \text{ arcsec} \quad \text{and} \quad \Delta\varphi = (1{,}61 \pm 0{,}31) \text{ arcsec}.$$

Later experiments of those type confirmed General Relativity to a higher precision.

ADVANCE OF THE PERIHELION OF MERCURY:
As known in astronomy for a long time and in contradiction to Newton's theory, the perihelion of mercury precesses around the sun by an angular velocity of

$$\omega = (43{,}11 \pm 0{,}45) \frac{\text{arcsec}}{\text{century}}.$$

General Relativity yields the theoretical value of

$$\omega = \frac{6\pi G_N M_\odot}{(1 - \varepsilon^2) a} = 43{,}04 \frac{\text{arcsec}}{\text{century}},$$

where ε is the excentricity and a the aphelion distance.

5.4 Gauge Fixing

ELECTRO-DYNAMICS:

The Maxwell equations for a vector potential A_α read

$$\Box A_\alpha - \partial_\alpha \partial_\beta A^\beta = -J_\alpha.$$

Those are four equations for four unknowns A_α, however the effective number of equations is only three, since the following identity always holds:

$$\partial^\alpha\left(\Box A_\alpha - \partial_\alpha \partial_\beta A^\beta\right) = \Box \partial^\alpha A_\alpha - \Box \partial_\beta A^\beta = 0.$$

This is, by the way, just current conservation $\partial_\alpha J^\alpha = 0$. One degree of freedom remains undetermined and this corresponds to the gauge invariance

$$A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x),$$

which is also a solution to the Maxwell equations for an arbitrary $\Lambda(x)$, since

$$\Box\, \partial_\alpha \Lambda - \partial_\alpha \partial_\beta \partial^\beta \Lambda = \Box\, \partial_\alpha \Lambda - \partial_\alpha \Box \Lambda = 0.$$

One may choose an arbitrary gauge condition as a fourth equation, for example the Lorentz gauge

$$\partial_\alpha A'^\alpha = 0,$$

which is reached by the gauge transformation

$$A'_\alpha = A_\alpha + \partial_\alpha \Lambda_L, \quad \Box \Lambda_L = -\partial_\alpha A^\alpha.$$

The latter equation can be solved for Λ_L and with this Λ_L the Maxwell equations become

$$\Box A'_\alpha - \left(\partial_\alpha \partial_\beta A^\beta + \partial_\alpha \partial_\beta \partial^\beta \Lambda_L\right) = \Box A'_\alpha \overset{!}{=} -J_\alpha.$$

EINSTEIN EQUATIONS:

Solving the Einstein equation means solving for $g_{\mu\nu}$, hence there are ten unknowns. There are also ten equations, however, as for the electro-dynamic case, differentiation w.r.t μ yields $T^{\mu\nu}{}_{;\mu} = 0$ due to energy-momentum conservation (4.9) and $G^{\mu\nu}{}_{;\mu} = 0$ due to the contracted Bianchi identity (4.5). So, effectively, there are only $10 - 4 = 6$ equations left.

As it turns out, the $g_{\mu\nu}(x)$-fields are determined by the Einstein equations only up to arbitray coordinate transformations

$$x'^\mu(x) = f^\mu(x),$$

equivalent to $A'_\alpha = A_\alpha + \partial_\alpha \Lambda$ in the electro-dynamic case. As was the Lorentz gauge, here the so-called *Harmonic gauge*

$$\boxed{\Gamma'^\lambda := g'^{\mu\nu}\, \Gamma'^\lambda{}_{\mu\nu} = 0}$$

is an elegant gauge condition. For a standard Minkowski coordinate system, this holds automatically, so for a weak gravitational field, the harmonic coordinate system is nearly Minkowskian.

WHY THIS GAUGE IS CALLED *HARMONIC*:

In mathematics, a function $f(x)$, $x \in \mathbb{R}^n$, is called *harmonic*, if it obeys the Laplace equation

$$\Delta f(x) = 0.$$

The four-vector generalization of the Laplacian is $\Box = \partial_\mu \partial^\mu$, whose covariant pendant is, applied to a function f, given by $g^{\mu\nu} f_{;\mu;\nu}$. Using those generalization, one may call f harmonic if

$$0 \overset{!}{=} g^{\mu\nu} f_{;\mu;\nu} = g^{\mu\nu} f_{,\mu;\nu} = g^{\mu\nu}\left(f_{,\mu,\nu} - \Gamma^\lambda{}_{\mu\nu} f_{,\lambda}\right) = \Box f - \Gamma^\lambda f_{,\lambda}.$$

Now, consider f to be x^α and use $\Box x^\alpha = \partial^\nu \partial_\nu x^\alpha = \partial^\nu \delta^\alpha_\nu = 0$. Thus, the coordinates x^α are *harmonic*, if

$$g^{\mu\nu} x^\alpha{}_{;\mu;\nu} = 0 \quad \Leftrightarrow \quad \Gamma^\lambda = 0.$$

Note, that f was not a vector, thus one can take only the α-component of x^α to equal f, which is neither a scalar nor a vector. Thus, this condition is not a tensor equation.

5.5 Einstein-Equation from Variational Principle

Let's postulate an action I which can be decomposed into a matter and a gravitational part,

$$I = I_M + I_G.$$

GRAVITATIONAL PART:

The action should be a scalar, thus an educated guess is

$$I_G = -\frac{1}{16\pi G}\int d^4x\, \sqrt{g} R.$$

To calculate the variation δI_G, consider

$$\delta\left(\sqrt{g}R\right) = \delta\left(\sqrt{g} g^{\mu\nu} R_{\mu\nu}\right)$$
$$= g^{\mu\nu} R_{\mu\nu} \delta\sqrt{g} + \sqrt{g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}.$$

FIRST TERM: For a matrix A with $a := \det A$, consider

$$a^{-1}\delta a = \delta \ln a = \delta \ln\det A = \ln\det(A + \delta A) - \ln\det A$$
$$= \ln\det(A + \delta A)/\det A = \ln\det\left(A^{-1}(A + \delta A)\right)$$
$$= \ln\det(1 + A^{-1}\delta A) \approx \ln(1 + \operatorname{Tr} A^{-1}\delta A) \approx \operatorname{Tr} A^{-1}\delta A.$$

Here, it was used that $A^{-1}\delta A \ll 1$. Analogous, it holds that

$$\frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \quad \Leftrightarrow \quad \delta\sqrt{g} = \frac{\delta g}{2\sqrt{g}} = \frac{1}{2}\sqrt{g} g^{\mu\nu}\delta g_{\mu\nu}.$$

SECOND TERM: Note that $g^{\kappa\mu} g_{\mu\nu} = \delta^\kappa_\nu$. Variation yields

$$g_{\mu\nu}\delta g^{\kappa\mu} + g^{\kappa\mu}\delta g_{\mu\nu} = 0 \qquad |\cdot g^{\mu\rho}$$
$$\Leftrightarrow \quad \delta g^{\kappa\rho} = -g^{\nu\rho} g^{\kappa\mu}\delta g_{\mu\nu}.$$

THIRD TERM: The third term turns out to vanish after integration with d^4x.

ALL TOGETHER: Thus, what remains is

$$\delta I_G = -\frac{1}{16\pi G}\int d^4x \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu}.$$

MATTER PART:

The variation of the matter part can be given as

$$\delta I_M = \frac{1}{2}\int d^4x\, \sqrt{g} T^{\mu\nu}\delta g_{\mu\nu},$$

which is a scalar, since $d^4x\, \sqrt{g}$ with $g := -\det g_{\mu\nu}$ is a scalar volume element and R is the Ricci scalar.

ALL TOGETHER:

Thus, from the variation principle, the condition for a extremal total action reads

$$-\frac{1}{16\pi G}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) + \frac{1}{2}T^{\mu\nu} = 0,$$

which is just equivalent to the Einstein equations.

6 The Schwarzschild Solution

6.1 Standard Form of the Metric

ASSUMPTIONS:

The Schwarzschild solution is a solution for a *static, isotropic* gravitational field, i.e. the field of a point mass. Thus, the metric should be independent of t and only depend on rotational invariants, which are

$$\vec{x}\cdot\vec{x},\quad \vec{x}\cdot d\vec{x},\quad d\vec{x}\cdot d\vec{x}.$$

Thus, the most general form of a spatially isotropic metric reads

$$ds^2 = -A\,dt^2 + B\,dt\,\vec{x}\,d\vec{x} + C(\vec{x}\,d\vec{x})^2 + D\,d\vec{x}^2,$$

where A, B, C, D are arbitrary function of r only.

DERIVE THE STANDARDFORM OF THE METRIC:

Adopting spherical coordinates $x^1 = r\sin\theta\cos\phi, x^2 = r\sin\theta\sin\phi, x^3 = r\cos\theta$ yields after some calculation

$$\vec{x}^2 = r^2,\quad \vec{x}\,d\vec{x} = r\,dr,\quad d\vec{x}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2.$$

Plugging this into the ds^2-formula and properly redefining A, B, C, D (e.g. absorb factors of r, r^2 into them) yields

$$ds^2 = -A\,dt^2 + B\,dt\,dr + C\,dr^2 + D(d\theta^2 + \sin^2\theta\,d\phi^2).$$

We now introduce new coordinates $t' = t, r'^2 = D(r), \theta' = \theta, \phi' = \phi$ and again redefine A, B, C (they are now functions of r') such that

$$ds^2 = -A\,dt^2 + B\,dt\,dr' + C\,dr'^2 + r'^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

Let us also introduce a new timelike coordinate t', defined by

$$dt' = \Phi\left(A\,dt - \frac{1}{2}B\,dr'\right)$$

$$\Leftrightarrow\quad A\,dt^2 - B\,dt\,dr' = \frac{1}{A\Phi^2}dt'^2 - \frac{B}{4A}dr'^2,$$

where $\Phi(t, r')$ is an integrating factor. The latter equation was obtained after squaring the former one. Defining new function $A' = 1/A\Phi^2$ and $B' = C + B/4A$ and omit all dashes yields

$$\boxed{ds^2 = -A\,dt^2 + B\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).}$$

6.2 Specific Form of the Metric

EINSTEIN EQUATIONS AND BOUNDARY CONDITIONS:

In vacuum, outside of the point mass, it is $T_{\mu\nu} = 0$ and according to 5.2 the Einstein equations read simply

$$R_{\mu\nu} = 0.$$

The boundary conditions read

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1,$$

since for $A = B = 1$ the Minkowski metric (in spherical coordinates) reads, as it should far away from the point mass.

FIND $\boldsymbol{A(r)}$ AND $\boldsymbol{B(r)}$:

From the boxed equation above, one can read out the metric

$$g_{\mu\nu} = \mathrm{diag}(-A, B, r^2, r^2\sin^2\theta)$$

and thus, calculate the affine connection thus the Ricci tensor (see Hobson p. 200 for results). As a result, it turns out that

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} = -\frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) \overset{!}{=} 0,$$

which must equal zero, since $R_{\mu\nu} = 0$ and thus $R_{00} = R_{11} = 0$. This is equivalent to

$$\Leftrightarrow\qquad 0 = A'B + AB' = (AB)' \qquad\Leftrightarrow\qquad AB = \text{const.} = 1$$

The constant must be 1 to obey the boundary conditions. Plugging in this result $B = 1/A$ into the R_{22} component yields

$$R_{22} = A - 1 + rA' = -1 + (rA)' \overset{!}{=} 0 \qquad\Leftrightarrow\qquad A = 1 + \frac{\text{const.}}{r}.$$

Note, that we found

$$-A = g_{00} = -1 - 2\phi_N = -1 + 2\,GM/r$$

as a Newtonian limit in 3.2. Apparently, the constant is $-2GM$. Altogether, the Schwarzschild solution (1916) reads

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).}$$

6.3 Singularities and the Schwarzschild Radius

BIRKHOFFS THEOREM:

Similar to Newtons theory, outside of a spherical body the metric is the same as for a point mass equal to the total mass of the body.

SINGULARITY AT THE SCHWARZSCHILD RADIUS:

Apparently, there is a singularity in the metric at the Schwarzschild radius

$$r_S = 2GM \overset{\text{SI}}{=} \frac{2GM}{c^2} \approx 2{,}95\ \text{km}\cdot\frac{M}{M_\odot},\qquad M_\odot = \text{mass of sun}.$$

First note, that for the sun it holds $R_\odot > r_S$, thus at $r = r_S$ the Schwarzschild matric is not applicable anyway. However, also for compact objects with a radius $R < r_S$, the *local* physical quantities (e.g. Riemann, Ricci tensor) are perfectly well behaved also around and at r_S. That is, $R_{\mu\nu}(r_S) = 0$ as it should. For example, one finds (Hobson, p. 200), using $B = 1/A$,

$$R_{22} = A - 1 + rA' = 1 - 2GM/r - 1 + r\,2GM/r^2 = 0,$$

which obviously holds for any $r \neq 0$. That is, the apparent singularity at $r = r_S$ is only a coordinate singularity and can be transformed away by an appropriate coordinate transformation.

SINGULARITY AT $\boldsymbol{r = 0}$:

In contrast to $r = r_S$, the singularity at $r = 0$ is real and cannot be get rid of by changing coordinates. The curvature at $r = 0$ is infinite.

GLOBAL SIGNIFICANCE OF THE SCHWARZSCHILD RADIUS:

Although *locally* everything is well behaved at r_S, *globally* r_S *is* special. Using 6.2 and $d\tau^2 = -ds^2$, one finds that

$$-ds^2 = d\tau^2 = \begin{cases} dt^2 - dr^2 + \cdots, & r\to\infty \\ (\text{pos})dt^2 + (\text{neg})dr^2 + \cdots, & r > r_S, \\ (\text{neg})dt^2 + (\text{pos})dr^2 + \cdots, & r < r_S \end{cases}$$

where (pos) means that this prefactor is positive. In the end, this is the reason, why nothing can escape the Schwarzschild radius.

HAWKING TEMPERATURE:

However, because of quantum effects, it is not true that nothing escapes the Schwarzschild radius or a black hole. A black hole is an almost perfect black body with the temperature, called *Hawking temperature* T_H given by

$$k_B T_H = \hbar c^3/8\pi GM \approx 10^{-7}\ \text{K}\cdot M_\odot/M.$$

6.4 Gullstrand–Painlevé Coordinates

Change the Schwarzschild coordinates (t, r, θ, ϕ) to (T, r, θ, ϕ), where

$$T = t + 4GM\left(\sqrt{\frac{r}{2GM}} + \frac{1}{2}\ln\left|\frac{\sqrt{r/2GM} - 1}{\sqrt{r/2GM} + 1}\right|\right).$$

After transformation, the metric in those new coordinates reads

$$ds^2 = -dT^2 + \left(dr + \sqrt{2GM/r}\,dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

which apparently is non-diagonal, but without any singularities at $r = 2GM$. In those coordinates, the so called *Kretschmann scalar* reads

$$K \coloneqq R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 12\frac{(2GM)^2}{r^6} = \frac{12}{r_S}\left(\frac{r_S}{r}\right)^6,$$

however, being a scalar, it is coordinate independent. Note also, that the singularity $r = 0$ appears again in K.

6.5 Generalization: The Kerr Solution

For a black hole with electric charge Q and intrinsic angular momentum S, the Kerr solutions reads

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\,d\varphi)^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2 + a^2)d\varphi - a\,dt\right)^2$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,$$

where

$$a \coloneqq S/M,\quad \Delta \coloneqq r^2 - 2Mr + a^2 + Q^2,\quad \rho^2 \coloneqq r^2 + a^2\cos^2\theta$$

7 Gravitational Waves

7.1 Magnitude of Gravitational Waves

Gravitational wave effects are extremely small. An atomic transition can emit a photon but also an electron. For the emission rate, it is $\Gamma_{\text{ph}} \sim \alpha \sim e^2$ and $\Gamma_{\text{gr}} \sim G$. Since $\Gamma_{\text{gr}}/\Gamma_{\text{ph}}$ is dimensionless, a factor of dimension energy squared is missing, which turns out to be the energy level difference:

$$\frac{\Gamma_{\text{gr}}}{\Gamma_{\text{ph}}} \sim \frac{G(\Delta E)^2}{e^2} \sim \frac{1}{\alpha}\left(\frac{\Delta E}{E_P}\right)^2 \sim 10^{-54}\left(\frac{\Delta E}{1\text{ eV}}\right)^2,$$

where $E_P = \sqrt{\hbar c^5/G}$ is the Planck energy. The rate for gravitation is so low that it will never be measured. However, gravitation is attractive only, so macroscopic sources can add up to measurable effects.

7.2 Weak Field Approximation and Harmonic Gauge

FIELD EQUATIONS IN WEAK FIELD APPROXIMATION:

The Einstein equations are non-linear, so to solve them we will adopt a weak field approximation, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad \eta_{\mu\nu} = \text{diag}(-1,1,1,1).$$

The affine connection and the Ricci tensor then read

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left(h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho}\right) + \mathcal{O}(h^2),$$

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\lambda}{}_{\lambda\mu,\nu} - \Gamma^{\lambda}{}_{\mu\nu,\lambda} + \mathcal{O}(h^2) \\ &= \frac{1}{2}\left(\Box h_{\mu\nu} - h^{\rho}{}_{\mu,\rho,\nu} - h^{\lambda}{}_{\nu,\mu,\lambda} + h^{\rho}{}_{\rho,\mu,\nu}\right) + \mathcal{O}(h^2), \end{aligned}$$

where it was used that, in first order of h, indices are raised and lowered with η. Using the Einstein equations in the form of 5.2 then gives

$$2R_{\mu\nu} = \Box h_{\mu\nu} - h^{\rho}{}_{\mu,\rho,\nu} - h^{\lambda}{}_{\nu,\mu,\lambda} + h^{\rho}{}_{\rho,\mu,\nu} = -16\pi G_N S_{\mu\nu},$$

where

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu, \quad S_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}.$$

The forces are assumed to be unimportant in the source and the ordinary conservation law $\partial_\mu T^{\mu}{}_{\nu} = 0$ holds.

CHOICE OF COORDINATES/HARMONIC GAUGE:

Consider a coordinate transformation

$$x'^{\mu} = x^{\mu} + \alpha^{\mu}(x),$$

where ε is similar small as h. Then, the metric transforms like

$$\begin{aligned} g'^{\mu\nu} &= \frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma} \\ \Leftrightarrow \quad \eta^{\mu\nu} + h'^{\mu\nu} &= \left(\delta^{\mu}_{\rho} + \alpha^{\mu}{}_{,\rho}\right)\left(\delta^{\nu}_{\sigma} + \alpha^{\nu}{}_{,\sigma}\right)\left(\eta^{\rho\sigma} + h^{\rho\sigma}\right) \\ \Leftrightarrow \quad h'^{\mu\nu} &= h^{\mu\nu} + \alpha^{\mu,\nu} + \alpha^{\nu,\mu} \end{aligned}$$

Since the Einstein equations are coordinate independent, h' also solves them, if h solves them. This gauge invariance is removed, for instance, by the harmonic gauge fixing condition (see 5.4)

$$g^{\mu\nu}\Gamma^{\lambda}{}_{\mu\nu} = 0,$$

which by plugging in Γ and g from above now reads

$$\begin{aligned} 0 = \eta^{\mu\nu}\frac{1}{2}\eta^{\lambda\rho}\left(h_{\nu\rho,\mu} + h_{\mu\rho,\nu} - h_{\mu\nu,\rho}\right) &= h^{\mu\lambda}{}_{,\mu} - \frac{1}{2}h^{\mu}{}_{\mu}{}^{,\lambda} \\ \Leftrightarrow \quad h^{\mu}{}_{\nu,\mu} &= \frac{1}{2}h^{\mu}{}_{\mu,\nu}. \end{aligned}$$

In this gauge the following terms in the field equations

$$-h^{\rho}{}_{\mu,\rho,\nu} - h^{\lambda}{}_{\nu,\mu,\lambda} = -\frac{1}{2}h^{\rho}{}_{\rho,\mu,\nu} - \frac{1}{2}h^{\lambda}{}_{\lambda,\mu,\nu} = -h^{\lambda}{}_{\lambda,\mu,\nu}$$

cancel out a third term and what remains is simply

$$\Box h_{\mu\nu} = -16\pi G_N S_{\mu\nu}.$$

7.3 Solution of the Weak Field Equations

GENERAL SOLUTION:

The Green's function $G^{\pm}$ of the $\Box$-operator, i.e.

$$\Box G^{\pm}(x,x') = -4\pi\delta^4(x-x'),$$

reads

$$G^{\pm}(x,x') = \frac{\delta\left(t' - (t \mp |\vec{x}-\vec{x}'|)\right)}{|\vec{x}-\vec{x}'|}.$$

Thus, the solution of the last equation in 7.2 reads

$$\begin{aligned} h_{\mu\nu}(\vec{x},t) &= 4G_N\int dx'\, S_{\mu\nu}(x')G^{\pm}(x,x') \\ &= 4G_N\int d^3\vec{x}'\,\frac{S_{\mu\nu}(\vec{x}',t\mp|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}, \end{aligned}$$

where only the time-integral was evaluated. The upper sign is for the retarded solution (where the mass configuration of the past creates the fields now), the lower sign the adcanced solution (where the mass configuration of the future creates the fields now). Obviously, only the former is implemented in nature.

PLANE WAVE SOLUTION IN VACUUM:

In vacuum, i.e. $S_{\mu\nu} = 0 \Longrightarrow \Box h_{\mu\nu} = 0$, the plane wave ansatz

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu}e^{ik_\sigma x^\sigma} + \varepsilon^{*}_{\mu\nu}e^{-ik_\sigma x^\sigma}$$

with constant polarization $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$ yields

$$\Box h_{\mu\nu} = -k_\rho k^\rho h_{\mu\nu} \overset{!}{=} 0 \quad \Longrightarrow \quad k^\rho k_\rho = 0.$$

Thus, gravitons have to be massless.

NUMBER OF INDEPENDENT POLARIZATIONS:

The gauge fixing condition from 7.2 yields

$$k_\mu\varepsilon^{\mu}{}_{\nu} = \frac{1}{2}k_\nu\varepsilon^{\mu}{}_{\mu}.$$

Those four equations fix four of the ten independent components of $\varepsilon_{\mu\nu}$. Furthermore, there is a so-called residual Gauge invariance. Consider the coordinate transformation

$$x'^{\mu} = x^{\mu} + i\alpha^{\mu}e^{ik_\nu x^\nu} - i\alpha^{*\mu}e^{-ik_\nu x^\nu} =: x^{\mu} + \eta^{\mu}.$$

From 7.2 we know, how $h_{\mu\nu}$ transforms, namely

$$h'_{\mu\nu} = h_{\mu\nu} + \eta_{\mu,\nu} + \eta_{\nu,\mu} = \left(\varepsilon_{\mu\nu} + k_\mu\alpha_\nu + k_\nu\alpha_\mu\right)e^{ik_\nu x^\nu} + \text{c.c.}$$

So obviously, the polarization transforms as

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu\alpha_\nu + k_\nu\alpha_\mu.$$

Here, we have four free parameters α_μ, which fix again four components. Since $\varepsilon'_{\mu\nu}$ still obeys the harmonic gauge condition, those four components can be fixed *in addition* to the four already fixed with the normal gauge condition. So we are left with $10 - 4 - 4 = 2$ independent polarizations. Let's proof that $\varepsilon'_{\mu\nu}$ indeed fulfills the harmonic gauge condition:

$$k_\mu\varepsilon'^{\mu}{}_{\nu} = \frac{1}{2}k_\nu\varepsilon'^{\mu}{}_{\mu}.$$

Plugging in the formula for $\varepsilon'^{\mu}{}_{\nu}$ and using $k^2 = 0$ yields

$$\begin{aligned} \Leftrightarrow \quad & k_\mu\varepsilon^{\mu}{}_{\nu} + k_\mu k^\mu\alpha_\nu + k_\mu k_\nu\alpha^\mu = \tfrac{1}{2}k_\nu\left(\varepsilon^{\mu}{}_{\mu} + k^\mu\alpha_\mu + k_\mu\alpha^\mu\right) \\ \Leftrightarrow \quad & k_\mu\varepsilon^{\mu}{}_{\nu} + k_\mu k_\nu\alpha^\mu = \tfrac{1}{2}k_\nu\varepsilon^{\mu}{}_{\mu} + k_\nu k^\mu\alpha_\mu \\ \Leftrightarrow \quad & k_\mu\varepsilon^{\mu}{}_{\nu} = \tfrac{1}{2}k_\nu\varepsilon^{\mu}{}_{\mu}. \end{aligned}$$

So, if ε fulfills the harmonic gauge, so does ε', too, always.

8 The Robertson-Walker-Metric

8.1 The Metric in Comoving Coordinates

THE METRIC:

The Robertson-Walker-Metric is a metric for a spatially homogenous and isotropic universe, i.e. a universe of an "smeared-out cosmic fluid". In comoving coordinates it reads

$$-ds^2 = d\tau^2 = dt^2 - a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right),$$

where $k \in \{0, \pm 1\}$ and $a \equiv a(t)$ is the *cosmic scale factor*. Here, t is the cosmic time, related to a scalar quantity, e.g. the temperature T_γ of the photons: $t \equiv t(T_\gamma)$. The coordinates r, θ, φ are *constant* got a comoving (freely falling) galaxy.

CURVATURE:

The curvature of the spatial subspace (3-space) reads

$$K^{(3)}(t) = \frac{k}{a^2}.$$

For $k = 0$, the 3-space is flat, for $k = \pm 1$ it has *constant* positive/negative curvature. It is finite for $k = 1$, but may be infinite for $k = 0, -1$ (it may still by finite, e.g. flat with periodic boundary conditions).

8.2 Current and Energy Momentum Tensor

DEFINITION:

For *comoving* galaxies, one can define a 4-current

$$J^\mu_{\text{gal}}(t) = n_{\text{gal}}(t) u^\mu, \quad u^\mu = (1, 0, 0, 0),$$

where n_{gal} is the number density of galaxies. The average cosmic matter has the energy momentum tensor

$$T_{\mu\nu} = \big(\rho(t) + p(t)\big) u_\mu u_\nu + g^{\text{RW}}_{\mu\nu} p(t).$$

For the Minkowski metric, this would read $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$, which hints at the fact, that ρ is something like the energy density and p the pressure.

CONSERVATION LAWS:

From the conservation laws $J^\mu{}_{;\mu} = T^{\mu\nu}{}_{;\nu} = 0$ it follows

$$n_{\text{gal}} a^3 = \text{const.}, \qquad \frac{d}{dt}(\rho a^3) + p \frac{d}{dt} a^3 = 0.$$

(For derivation see exercise 12.2)

The first equation is simply particle number conservation. The second equation is analogous to $dQ = dU + dW$ from thermodynamics with $U \mathrel{\widehat{=}} \rho a^3$ and $dW = p \, dV \mathrel{\widehat{=}} p \, \partial_t a^3$. Thus, the second equation means that there is no energy flow out of or into the universe.

8.3 Hubble-Redshift

REDSHIFT DUE TO EXPANSION:

Set the earth (observer) into the origin of the coordinate system and consider light being emitted by a comoving galaxy (r, θ, φ fixed). For the light travelling radially to earth, it holds $d\theta = d\varphi = 0$ and thus

$$d\tau^2 = 0 = dt^2 - a^2 \frac{dr^2}{1 - kr^2}.$$

The first wave crest is emitted at t_1 and observed at t_0. Integrating the above equation, taking into account that the light travels in $-r$-direction, yields

$$I_k(r_1) \coloneqq -\int_{r_1}^{0} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t_1}^{t_0} \frac{dt}{a}.$$

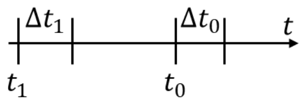
The next wave crest then is emitted about Δt_1 and received about Δt_0 later:

$$I_k(r_1) = \int_{t_1 + \Delta t_1}^{t_0 + \Delta t_0} \frac{dt}{a}.$$

Subtracting those two equations yields

$$0 = \int_{t_1 + \Delta t_1}^{t_0 + \Delta t_0} \frac{dt}{a} - \int_{t_1}^{t_0} \frac{dt}{a} = \int_{t_0}^{t_0 + \Delta t_0} \frac{dt}{a} - \int_{t_1}^{t_1 + \Delta t_1} \frac{dt}{a},$$

which can be made visual by the following sketch:



If Δt_i is so small that a doen't vary much during this time, the integral can be evaluated as a product:

$$0 \approx \frac{\Delta t_0}{a(t_0)} - \frac{\Delta t_1}{a(t_1)} \quad \Leftrightarrow \quad \frac{\nu_0}{\nu_1} = \frac{\Delta t_1}{\Delta t_0} \approx \frac{a(t_1)}{a(t_0)},$$

where ν_i are the frequencies of the light. Since $t_1 < t_0$, for an expanding universe (i.e. $a(t_1) < a(t_0)$), we observe redshifts, $\nu_0 < \nu_1$.

HUBBLE'S LAW:

The redshift z reads ($\lambda = c/\nu$)

$$z \coloneqq \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\nu_1}{\nu_0} - 1 \approx \frac{a(t_0)}{a(t_1)} - 1.$$

For a not too far away galaxy, $a(t_1)$ may be Taylor expanded around t_0:

$$z \approx \frac{a(t_0)}{a(t_0) + \dot{a}(t_0)(t_1 - t_0)} - 1 = \frac{1}{1 + \frac{\dot{a}(t_0)}{a(t_0)}(t_1 - t_0)} - 1$$

$$\approx 1 - \frac{\dot{a}(t_0)}{a(t_0)}(t_1 - t_0) - 1 = \frac{\dot{a}(t_0)}{a(t_0)} \frac{r_1}{c} = H_0 \frac{r_1}{c}.$$

Here it was used, that also $t_1 - t_0$ is small and $t_1 - t_0 = -r_1/c$. This is Hubble's law.

9 The Expanding Universe

9.1 The Friedmann Equations

THE REDUCED FIELD EQUATIONS:

The Robertson-Walker-Metric had components

$$g_{tt} = -1, \quad g_{it} = 0, \quad g_{ij} = a^2 \tilde{g}_{ij}(\vec{x}),$$

where the $\tilde{g}_{ij}$ can be read off from 8.1. The Einstein field equations from 5.2 read

$$R_{\mu\nu} = -8\pi G_N S_{\mu\nu}, \quad S_{\mu\nu} := T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda.$$

Using the Robertson-Walker-Metric, one will find

$$R_{tt} = 3\,\ddot{a}/a, \quad R_{it} = 0, \quad R_{ij} = -(a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij}.$$

Taking the energy-momentum tensor from 8.2 yields

$$T^\lambda{}_\lambda = g^{\mu\nu} T_{\mu\nu} = (\rho + p) g^{00} + p\delta^\mu{}_\mu = 3p - \rho,$$

where $g^{00} = -1$, $\delta^\mu{}_\mu = 4$. Thus, $S_{\mu\nu}$ reads

$$S_{tt} = \frac{1}{2}(\rho + 3p), \quad S_{it} = 0, \quad S_{ij} = \frac{1}{2} a^2 \tilde{g}_{ij}(\rho - p).$$

Thus, the reduced (by the Ansatz) field equations read

$$3\ddot{a} = -4\pi G(\rho + 3p)a, \qquad a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\rho - p)a^2.$$

THE FRIEDMANN EQUATIONS:

Multiplying the first of the above equations with a and subtract three times the second equation gives the first Friedmann equation:

$$F_I \quad :\Leftrightarrow \quad \dot{a}^2 + k = \frac{8\pi G}{3} \rho a^2.$$

For the second Friedmann, recall $\partial_t \rho a^3 + p\partial_t a^3 = 0$ from 8.2. Multiplying this by dt/da yields

$$F_{II} \quad :\Leftrightarrow \quad \frac{d}{da} \rho a^3 = -3pa^2.$$

We have three unknowns a, p, ρ; thus, we need a third equation, e.g. the equation of state:

$$F_{III} \quad :\Leftrightarrow \quad p = p(\rho) = \begin{cases} 0, & \text{non-relativistic matter} \\ \rho/3, & \text{ultra-relativistic matter} \end{cases}.$$

9.2 Qualitative Insights from the Friedmann Equations

RADIANT WAS DOMINANT IN EARLY UNIVERSE:

The Friedmann equation F_{II} gives for non- and ultra-relativistic matter

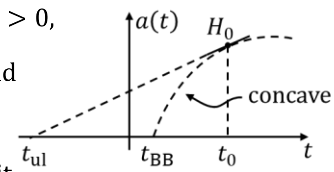
$$F_{II} \quad \Rightarrow \quad \begin{cases} \text{non-rel.:} & \partial_a \rho a^3 = 0 & \Rightarrow \quad \rho \sim a^{-3} \\ \text{ultra-rel.:} & \partial_a \rho a^3 = -\rho a^2 & \Rightarrow \quad \rho \sim a^{-4} \end{cases}.$$

Hence, *if* at present time we have $0 < \rho_{\text{u-rel.}} \ll \rho_{\text{n-rel.}}$ then, since a is increasing, there must have been a time, where $\rho_{\text{u-rel.}} > \rho_{\text{n-rel.}}$, i.e. radiation was dominant in the early universe, i.e. we had a *hot* big bang.

THERE WAS A BIG BANG, I.E. $a = 0$:

Recall the first reduced field equation from 9.1. From $a \geq 0$ and $\rho + 3p \geq 0$ (which holds for all *known* matter), we have $\ddot{a} \leq 0$. Thus, $a(t)$ is concave and since the present ($t = t_0$) expansion rate is positive,

$$H_0 := \left.\frac{\dot{a}}{a}\right|_{t_0} > 0,$$



there must have been a big bang and some time t_{BB}, where $a(t_{BB}) = 0$.

AGE OF THE UNIVERSE:

Since $a(t)$ is concave, an upper limit t_{ul} can be set on the age of the universe, namely

$$t_{\text{ul}} = H_0^{-1} \approx 13 \cdot 10^9 \text{ yr}.$$

CRITICAL DENSITY:

At present time, F_I is equivalent to

$$\rho_0 = \frac{3}{8\pi G}\left(H_0^2 + \frac{k}{a_0^2}\right) =: \rho_{0c} + \frac{3k}{8\pi G a_0^2}, \quad \rho_{0c} = \frac{3H_0^2}{8\pi G}.$$

Obviously, $\rho_0 \lesseqqgtr \rho_{0c} \Rightarrow k \lesseqqgtr 0$. We have measured that

$$\Omega := \rho_0/\rho_{0c} \quad \Rightarrow \quad 0{,}1 \leq \Omega_m \leq 0{,}2.$$

Ω_m takes all matter (incl. dark matter) density into account.

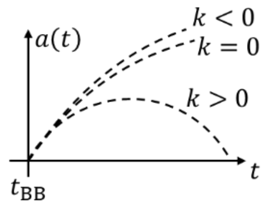
9.3 More Detailed Behavior of the Scale Factor

MATTER DOMINATED UNIVERSE:

Let's assume, $k = 0$. In the matter dominated universe ($p = 0$), from 9.2 or F_{II} in 9.1 it is known that $\rho \sim a^{-3}$. Hence,

$$F_I \quad \Rightarrow \quad \dot{a}^2 \sim \rho a^2 \sim a^{-1} \quad \Rightarrow \quad a \sim t^{2/3}.$$

For $k \neq 0$, solutions can also be found: The solution for $k > 0$ is qualitatively different from the others; here there is a big crunch.



So, choosing $t_{BB} = 0$, we may write

$$a(t) = a_0 (t/t_0)^{2/3}$$

and hence $a(t_{BB}) = 0$ as well as

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \quad \Rightarrow \quad t_0 = \frac{2}{3H_0} \approx 8{,}7 \cdot 10^9 \text{ yr}.$$

Thus, from the behavior $a \sim t^{2/3}$, the age of the universe is 2/3 of its maximum possible value from 9.2. The problem is: This is younger that observed stars.

RADIATION DOMINATED UNIVERSE:

For early time, we can neglect the k-term in F_I as will be justified a posteriori. From F_{II} we had in 9.2 already $\rho \sim a^{-4}$, hence

$$F_I \quad \Rightarrow \quad \dot{a}^2 = \frac{8\pi G}{3} \rho a^2 \sim a^{-2} \quad \Rightarrow \quad a \sim t^{1/2}.$$

By the Stefan-Boltzmann law $\rho \sim T^4$, we have $T \sim a^{-1}$. It follows that $\dot{a}^2 \sim t^{-1}$ and $\rho a^2 \sim a^{-2} \sim t^{-1}$, whereas $k \sim t^0$. Thus, for early times, the k-term could be neglected.

9.4 Accelerating Universe: Vacuum Energy

THE COSMOLOGICAL CONSTANT:

In 1917 Einstein proposed a generalization for his field equations (see 5.1) to make a static universe possible by introducing a constant λ :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G\left(T_{\mu\nu} - \lambda g_{\mu\nu}\right) =: -8\pi G \tilde{T}_{\mu\nu}.$$

For a perfect fluid, like in 8.2, the energy-momentum tensor takes the form

$$\tilde{T}_{\mu\nu} = (\rho + p) u_\mu u_\nu + g_{\mu\nu} p - \lambda g_{\mu\nu} =: (\tilde{\rho} + \tilde{p}) u_\mu u_\nu + g_{\mu\nu} \tilde{p},$$

which defines

$$\tilde{\rho} := \rho + \lambda, \qquad \tilde{p} = p - \lambda.$$

For $\lambda > 0$, the "vacuum energy density" increases the energy density but reduces the pressure. By replacing $\rho \to \tilde{\rho}$ and $p \to \tilde{p}$ all the previous results remain valid. The first Friedmann equation becomes

$$\tilde{F}_I \quad :\Leftrightarrow \quad \dot{a}^2 + k = \frac{8\pi G}{3} \tilde{\rho} a^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda}{3} a^2$$

with the cosmological constant

$$\Lambda = 8\pi G \lambda.$$

THE STATIC EINSTEIN-UNIVERSE:

For a static universe ($\dot{a} = 0$) the Friedmann equations read

$$\tilde{F}_I: \quad k = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda}{3} a^2,$$

From $\tilde{\rho} = -3\tilde{p}$ (don't know where this comes from) follows $\rho = 2\lambda$. Plugging this into $\tilde{F}_I$ yields

$$a = \sqrt{k/\Lambda} \qquad \text{and of course} \qquad \rho = \frac{\Lambda}{4\pi G}.$$